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ANISOTROPIC DAMAGE MECHANICS BASED ON STRAIN ENERGY EQUIVALENCE AND EQUIVALENT ELLIPTICAL MICROCRACKS

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Abstract—A theory of damage mechanics is introduced based on a principle of strain energy equivalence. This principle is used to develop the effective continuum elastic properties of a damaged solid in terms of the undamaged elastic properties and a scalar damage field. The damage variable is defined as the volume fraction of a damage zone associated with equivalent elliptical microcracks. This definition provides a means by which a damaged isotropic material can exhibit anisotropic (orthotropic) properties, and entails determining effective crack orientation and geometry factors from the local deformation. Strain energy dissipation associated with crack growth (not nucleation) is used to develop a consistent damage evolution equation. This evolution equation is related to the standard power law model of crack growth commonly used in fracture mechanics, and to the equivalent stress measure commonly used in mechanics of plastic deformation. The combination of representing local damage as an effective elliptical crack volume fraction, a consistent damage evolution equation, and the determination of effective elastic properties using a strain energy equivalence principle yields a simple, yet powerful, approach to predicting failure of mechanical components. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

In the half-century since the end of World War II, extensive work has addressed the characterization of failure modes of various systems: mechanical, electronic, and most recently, software. The ability to accurately predict the remaining operational life of a mechanical system is an essential element of a condition-based maintenance approach (Hansen *et al.* 1995). Numerous approaches to the characterization and prediction of mechanical failure modes and damage evolution have appeared in the literature associated with various fields, including physics, applied mathematics, material sciences and engineering, fracture mechanics, and damage mechanics.

Weibull (1951) proposed a statistical distribution function based on a weakest-link theory so as to predict the fatigue life of a brittle structure. This theory was improved by Hunt and McCartney (1979) to account for the nature of a defect distribution. Their theory was in turn modified by Hu *et al.* (1988) by considering crack growth characteristics. In this literature, the failure probability $F(\sigma, t)$ at an applied stress σ is given by

$$F(\sigma, t) = 1 - e^{-U(\sigma, t)} \tag{1}$$

where the quantity U in eqn (1) describes the probability of finding a defect having size larger than $a(\sigma)$ which may initiate component failure at an applied stress σ below the ultimate tensile strength of the material.

In 1989, Voigt proposed a simple relation to predict rate-dependent material failure:

$$\dot{\Omega}^{-\alpha}\ddot{\Omega}-A=0 \tag{2}$$

In eqn (2), Ω is a measurable quantity such as strain and A and α are empirical constants. Voigt suggested that this equation might be applicable to various branches of engineering, material science, and the earth sciences. He also showed that, in a special case, eqn (2) can be reduced to a form of the well-known Kachanov–Rabotnov equation (Kachanov, 1958; Rabotnov, 1969) as follows:

$$\dot{D}=\frac{A\sigma^p}{(1-D)^q} \tag{3}$$

where σ is the applied stress and A, p, and q are empirical material constants. In eqn (3), the damage variable is defined as $0 \leqslant D \leqslant 1$. The case $D = 0$ corresponds to the undamaged state, while $D = 1$ is equivalent to complete local rupture of the material. Experiments have shown that final component failure is initiated at a critical value of D well below 1. It usually takes on values between 0.2 and 0.8 depending on the material (Lemaitre, 1985, 1990). Since Rabotnov (1969) first introduced the concept of effective stress $\bar{\sigma} = \sigma/(1-D)$ using eqn (3) with $p = q$, numerous researchers have adopted this concept in the development of various damage models. Extensive treatments of continuum damage mechanics can be found in the books by Kachanov (1986) and Lemaitre (1992), and also in the paper by Krajcinovic (1989). Additional review of current theories of damage mechanics is presented later in this paper.

In the field of fracture mechanics, much material failure prediction has been accomplished using the Paris law (Paris and Erdogan, 1963) for cyclic loading, and Evan's equation (Evan, 1973) for slow crack growth, given as

$$\dot{a} = AK^n \tag{4}$$

where $\dot{a}$ is the crack tip velocity, K is the crack tip stress intensity factor, and A, n are constants appropriate to the particular material–environment system.

It is evident from this brief review that there is not a unique method for prediction of damage evolution and material failure and, furthermore, that one model may not be easily related to other models.

A material failure process is often assumed to involve general degradation of elastic properties due to the highly localized nucleation and growth of microdefects (i.e., microcracks and microvoids) and their ultimate coalescence into macrodefects. The process and result of these irreversible, energy dissipating, microstructural rearrangements are often called damage. Because of the complex nature of damage, there is no general agreement regarding the definition of damage variable(s). As Krajcinovic and Mastilovic (1995) discussed, selection of a damage variable is largely a matter of taste and convenience, and often has no obvious physical basis.

Despite the non-uniqueness of damage definitions, much research has addressed the two additional major subjects of damage mechanics: constitutive equations of damaged materials, and damage evolution laws. The larger portion of the (continuum) damage mechanics literature is probably devoted to the development of damage variables and constitutive equations rather than to the development of damage evolution (or kinetic) equations, much less to the solution of boundary value problems. Table 1 classifies some damage mechanics models for isotropic materials on the basis of: (a) the type of elastic behavior of the damaged material; and (b) the scalar or tensor nature of the damage variables and evolution equations. As shown in the table, one of the goals of the present work is to develop a model that permits anisotropic behavior of the damaged material, while using a scalar damage variable.

Since damage is assumed to degrade, at least locally, the elastic properties of a material, response modeling must address the formulation of constitutive properties, a task which may be approached using micromechanical and/or phenomenological approaches. An

Table 1. Classification of damage mechanics theories for initially isotropic materials

Types of models	Damage Variable/Damage Evolution Law: ISOTROPIC (scalar)	ANISOTROPIC (tensor)
Equivalent Elastic Moduli (Constitutive Law of Damaged Material)		
Isotropic	Kachanov (1958, 1986) Rabotnov (1969) Hayhurst, Leckie (1973) Leckie, Hayhurst (1977) Lemaitre, Chaboche (1974) Leckie (1978) Lemaitre, Plumtree (1979) Lemaitre (1985, 1986, 1990, 1992) Cocks, Leckie (1987) Simo, Ju (1987) Fotiu *et al.* (1991)	Davison, Stevens (1973) Murakami, Ohno (1981) Murakami (1988)
Anisotropic	Lee, Lesieutre, Fang (1997, present paper)	Vakulenko, Kachanov (1971) Sidoroff (1980) Krajcinovic, Fonseca (1981) Krajcinovic (1983a, b, 1985, 1989) Chow, Wang (1987) Yazdani (1993) Lubarda *et al.* (1994)

extensive review of such approaches can be found in Krajcinovic (1989). The micromechanical modeling process leads to a one-to-one correspondence between a discontinuous field on an inhomogeneous mesoscale and an effective continuous field on the homogenous macroscale. The homogenization (averaging) of the mesostructural field of defects within a representative volume element (RVE) into a macrofield of the effective continuum corresponds to micromechanical modeling.

Since an RVE is assumed, by definition, to contain a statistically sufficient (large) number of random microcracks (permitting a reduction of computational effort), the self-consistent method has emerged as one of the simplest effective continuum modeling methods. This method has been employed by Budiansky and O'Connell (1976), followed by Horri and Nemat-Nasser (1983), Sumarac and Krajcinovic (1987) and Krajcinovic and Sumarac (1989). Despite clarity and a well-defined relationship with physical phenomena, micromechanical models based on the methods of elasticity and fracture mechanics in conjunction with specific realizations of stochastic defects within an RVE require significant computational resources to determine the internal details of the response within an RVE. Furthermore, it may be impractical or impossible to accurately predict and/or to experimentally measure the microcrack density parameters that are usually required in the self-consistent methods (i.e., the number of microcracks in a RVE and their geometries), especially during the phases of crack generation and growth.

In contrast to micromechanical models, phenomenological models do not consider the micro-details of material response, but describe damage indirectly by introducing internal (or hidden) variables. This has caused some confusion and spawned more extensive, substantially different, models of the same phenomena. Since the selection of the damage variable is perhaps the most important step, irreversible thermodynamics (see Kestin and Bataille, 1977, and Ziegler, 1983) has been used to provide a scientific basis for theories of continuum damage mechanics (Murakami and Ohno, 1981; Krajcinovic and Fonseka, 1981; Krajcinovic, 1983a,b, 1985; Lemaitre, 1985; Cocks and Lecki, 1987; Simo and Ju, 1987; Chow and Wang, 1987; Fotiu *et al.*, 1991).

Historically, the Kachanov–Rabotnov equation (eqn 3), attributable originally to Kachanov (1958), provided a basis for Rabotnov's effective stress concept (1969) and later for Lemaitre and Chaboche's effective stiffness concept (1975). Based on this interpretation of damage, many researchers have focused on generalizing the 1-D constitutive equation of a damaged material to anisotropic damage states induced by a three-dimensional distribution of defects. Such research includes scalar damage variables (Kachanov, 1958; Lemaitre, 1985), vector variables (Davison and Stevens, 1973; Krajcinovic and Fonseka, 1981; Krajcinovic, 1985), second order tensors (Vakulenko and Kachanov, 1971; Murakami and Ohno, 1981), and fourth order tensors (Chaboche, 1979; Sidoroff, 1981; Simo and Ju, 1987; Chow and Wang, 1987; Lubarda and Krajcinovic, 1993; Yazdani, 1993). Despite these developments, the loss of physical insight, the complexity of mathematical formulation, and the practical difficulties of measuring damage parameters restrict the applicability of many damage definitions available in the literature. Despite extensive research, the appropriate definition of damage variable(s) and the development of corresponding evolution equations and constitutive equations still seem to be open issues.

The Kachanov–Rabotnov damage evolution equation was the first of its kind. There have since been continuous efforts to modify and extend it (e.g., Hayhurst and Leckie, 1973; Lemaitre and Chaboche, 1975; Leckie and Hayhurst, 1977; Leckie, 1978; Lemaitre and Plumtree, 1979; Kachanov, 1986). In addition, there have been rigorous efforts to derive damage evolution laws based on irreversible thermomechanics (Cailletaud *et al.*, 1981; Murakami and Ohno, 1981; Lemaitre, 1985, Simo and Ju, 1987; Chow and Wang, 1987; Cocks, 1987; Yazdani, 1993; Krajcinovic, 1983a, b, 1985, 1992; Fotiu *et al.*, 1991; Maugin, 1992).

Also note that, in many papers, constitutive equations and damage evolution equations are developed by replacing the Cauchy stress of the undamaged material with the effective stress on the basis of the strain equivalence principle introduced by Lemaitre (1985, 1992). The use of this principle generally increases the complexity of the governing equations; this may have motivated the introduction of the alternative stress equivalence principle by Simo and Ju (1987). Although the effective stress or strain concepts can be converted to an effective stiffness concept via the strain or stress equivalence principles, and vice versa, choosing and consistently applying one of these principles can be confusing. An effective stiffness principle has the apparent advantage that the governing equations retain their nominal form. Sidoroff (1981) pointed out that an effective stiffness approach is equivalent to an effective stress approach only if an effective strain is used.

Thus, the purposes of this paper are: (1) to introduce a strain energy equivalence principle as an alternative to the available strain- or stress-equivalence principles; (2) to introduce the concept of an equivalent elliptical crack representation of local damage; (3) to derive the equivalent continuum constitutive laws of a damaged material in terms of the undamaged isotropic material properties and a new damage variable; and (4) to derive a consistent damage evolution equation from the damage variable definition and the fracture mechanics crack growth law. One feature of the combined result is that damage growth and propagation is consistent with fracture mechanics.

2. CLASSICAL CONTINUUM DAMAGE MECHANICS

Among the many definitions of damage variables, those related to changes in material elastic moduli seem to be most popular. For example, in case of a scalar damage variable D, and one-dimensional stress, the effective Young's modulus for the damaged material, $\bar{E}$, is expressed in terms of that of the undamaged material, E, as follows:

$$\bar{E} = E(1-D). \tag{5}$$

The constitutive equation for the damaged material shows that

$$\varepsilon = \frac{\sigma}{\bar{E}} = \frac{\sigma}{E(1-D)} = \frac{\bar{\sigma}}{E} \tag{6}$$

where σ and ε are the nominal stress and strain, respectively. Equation (5) suggests that the

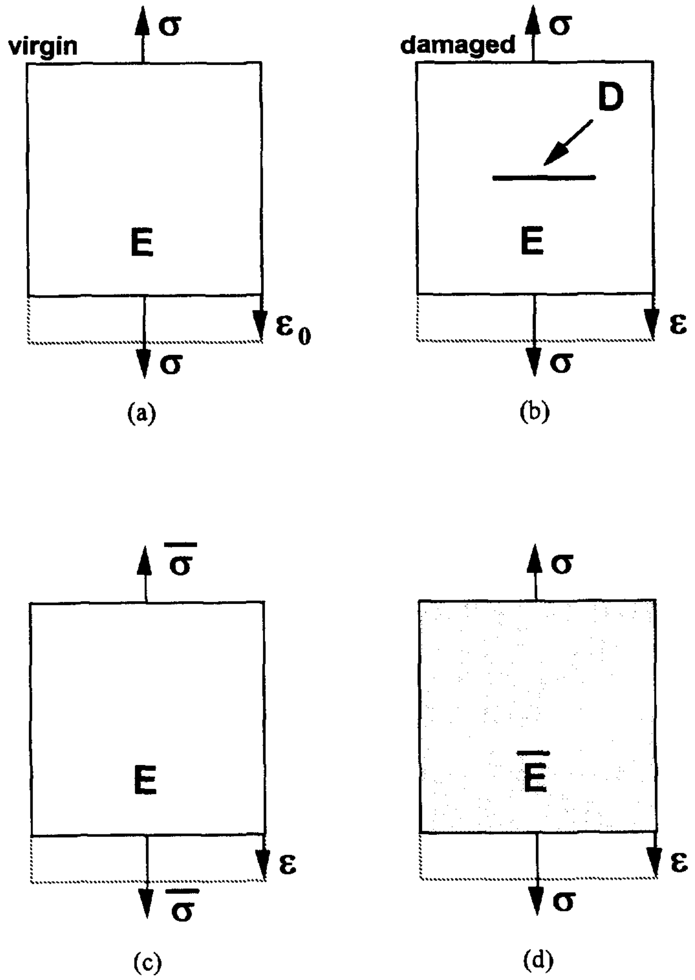

Fig. 1. Uniaxial illustration of damage mechanics concepts: (a) virgin (undamaged) state; (b) current damaged state; (c) damage represented by the effective stress, $\bar{\sigma}$, based on the strain equivalence principle; (d) damage represented by the effective modulus, $\bar{E}$, based on strain energy equivalence principle.

nominal stress σ can be replaced by the effective stress $\bar{\sigma} = \sigma/(1-D)$ in the damaged material under the same strain. Based on this observation, Lemaitre (1985) proposed the strain equivalence principle, stating that the strain behavior of a damaged material may be represented by the constitutive equations of the virgin material, and simply replacing the stress by the effective stress. Figure 1 illustrates some basic concepts of the strain equivalence principle (Fig. 1(c)), as well as other approaches to representing the behavior of material with damage.

Since the effective undamaged states shown in Figs 1(c) and 1(d) are equivalent in the sense that they represent the same damage, the two corresponding homogenized models should have the same strain energy density as the actual damaged material shown in Fig. 1(b). However, the damage model corresponding to Fig. 1(c) has the strain energy $V = \bar{\sigma}\varepsilon/2 = E\sigma^2/2$ (since $\bar{\sigma} = E\varepsilon$ in this case) while the damage model corresponding to Fig. 1(d) has $V = \sigma\varepsilon/2 = E\varepsilon^2/2$ (since $\sigma = \bar{E}\varepsilon$ in this case). This inconsistency shows that the concepts of effective stress and effective stiffness are not convertible without care and restriction. Hence, the strain equivalence principle is considered as just one possible principle in continuum damage mechanics (Lemaitre, 1992). As noted in the preceding, replacing the nominal stress with the effective stress (in the framework of the strain equivalence principle) or the nominal strain with the effective strain (in the framework of the stress equivalence principle) generally increases the complexity of the governing equations. The possibility of developing simpler, more straightforward principles for use in continuum damage mechanics seems a reasonable one to explore.

3. STRAIN ENERGY EQUIVALENCE PRINCIPLE

In the structural dynamics community, equivalent continuum modeling (ECM) approaches have proven capable of capturing the global behavior of periodic lattice structures, among others. Many of these approaches have been based on energy equivalence concepts (Noor *et al.*, 1978; Lee, 1990, 1994). In this context *energy equivalence* means that the original structure and its equivalent continuum model must contain equal kinetic and strain energies when both are subject to the same global displacement and velocity fields.

At the micro-level, damaged materials with microdefects may be regarded as discontinuous. The development of a continuum representation of the mechanical behavior of such a material is one of the goals of the field of damage mechanics.

In the development of an equivalent continuum model of a damaged material, it is assumed that the distances between small microdefects are sufficiently large so that each defect is affected only by the stress distribution around the microcrack. Then, a small material volume cell (MVC) that contains only a single defect can be isolated as shown in Fig. 2. Since an MVC contains only one microcrack, it differs substantially from an RVE, which is generally assumed to contain a large number of microcracks, the statistical distribution of which is known (Hill, 1963; Krajcinovic, 1989). Since the material within an MVC is assumed to be homogeneous, its geometrical scale is evidently larger than that of an RVE.

The effective properties of the material in a uniform ECM volume cell are to be determined, as illustrated in Fig. 2. The strain on the boundary of the MVC of the damaged material is taken to be same as that on the ECM. This assumption implies that the macrobehavior represented by the ECM is the same as that of the damaged material. From this observation, the strain energy equivalence principle may be introduced. This principle will be used to develop the effective continuum elastic properties of the damaged material as well as a new definition of damage variable.

Principle of Strain Energy Equivalence: When the MVC of the damaged material and its ECM volume cell have identical global displacements on their boundaries, they contain equal strain energy.

Although this principle might be extended to include kinetic energy in an attempt to capture the internal dynamics associated with rapid damage growth, the discussion herein is limited to slowly growing damage. Figure 3 illustrates the general concept of the damage modeling procedure based on this strain energy equivalence principle (SEEP). The SEEP provides the effective continuum elastic properties of the ECM by equating the strain energy

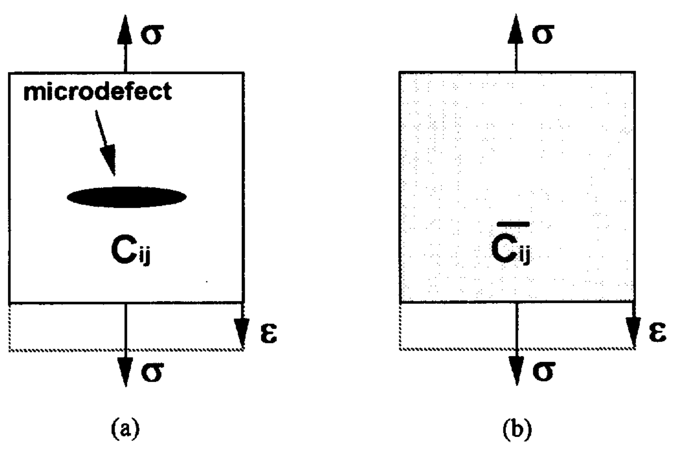

Fig. 2. (a) Material volume cell (MVC) containing a single microdefect; and (b) its effective continuum model (ECM).

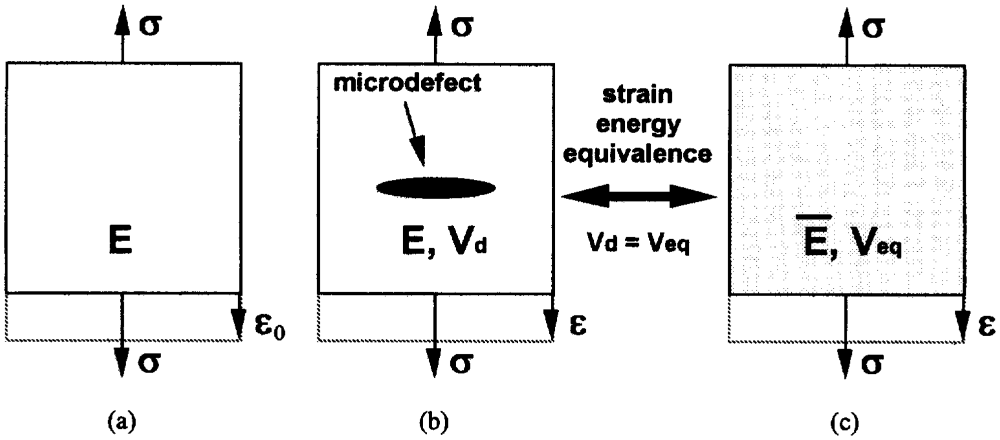

Fig. 3. Damage mechanics based on the strain energy equivalence principle: (a) virgin (undamaged) state; (b) current damaged state; and (c) its effective continuum representation.

density V_d contained in the MVC of the damaged material to the strain energy density V_{eq} in the corresponding ECM. That is

$$V_{eq}(\bar{\mathbf{C}};\varepsilon) = V_d(\mathbf{C},\mathscr{D};\varepsilon) \tag{7}$$

where $\mathbf{C} = C_{ij}$ represents the elastic stiffness of the undamaged material, $\bar{\mathbf{C}} = \bar{C}_{ij}$ the effective continuum elastic stiffness, $\mathscr{D}$ the damage variable consistent with $\bar{\mathbf{C}}$, and ε represents the average strain on the boundaries of the MVC and ECM. Thus, complete constitutive equations for coupled elasticity and damage may be obtained by replacing the undamaged elastic stiffness with the effective continuum elastic stiffness $\bar{\mathbf{C}}$, calculated from eqn (7), without redefining or changing the nominal stress or strain appearing in the original constitutive equations. This postulate results in straightforward formulation of the governing equations of damaged material behavior. Furthermore, it seems consistent with the physical interpretation of elastic modulus degradation due to reduction in the effective stress-transmitting area (Lubarda *et al.* 1994).

In eqn (7), the strain energy density V_d for the damaged material with defined microcracks may be calculated using conventional stress analysis. Alternatively, fracture mechanics provides closed-form solutions for certain crack problems in both two- and three-dimensional solids that can be used as a starting point for determining effective properties. Initial results are obtained for an elliptical through-crack in a two dimensional solid, with additional effort focused on generalizing the damage theory results to three dimensional solids.

In the damage mechanics literature, Chow and Wang (1987) seem to be the first to use the terminology "elastic energy equivalence". In deriving an effective stiffness tensor in terms of an anisotropic damage variable, however, they used the effective stress concept. The "fictitious" elastic energy equation for the damaged body was developed by simply replacing the nominal stress with the effective stresses of the undamaged body. Hence, the damage theory based on the strain energy equivalence principle introduced herein and discussed in the following sections, is distinct from their work.

4. EFFECTIVE CONTINUUM ELASTIC PROPERTIES AND DAMAGE VARIABLES

To develop a damage model based on SEEP, the change of the strain energy storage capacity of an elastic body due to the presence of microdefects is considered. A damaged body cannot store as much strain energy under a given deformation as an undamaged body because of the degradation of elastic moduli. Since any microdefect is assumed to reduce the effective elastic moduli, various kinds of microdefects (or damage in general sense) may be considered as equivalent microcracks. Under this postulate, the development focuses on damage associated with such microcracks.

In fracture mechanics, the change in strain energy associated with forming new surfaces in a body has been explored over many decades. The strain energy released in forming a crack is often called the crack energy. Crack energies for elliptical through cracks in an infinite two-dimensional elastic body and for elliptical surface cracks embedded in an infinite three-dimensional elastic body have been calculated by Kassir and Sih (1967, 1975) and Sih and Leibowitz (1968). In order to apply their results to a MVC with a microcrack, the microcrack size is assumed to be relatively small compared to the characteristic length of a MVC, so that the crack energy in a MVC may be approximated by that in an infinite body. This approximation is appropriate in that the effects of neighboring cracks decay rapidly with distance. For example, in the case of an infinite periodic array of equally spaced cracks in an infinite plate, the interaction decreases in proportion to the second power of the ratio of the crack size to the distance between neighboring cracks (Rice, 1968). The validity of this approximation is also suggested by the fact that complete local failure of material generally occurs before the damage variable D in eqn (5) reaches the value of 1 that implies absence of material (Lemaitre, 1985, 1990; Krajcinovic, 1989).

As shown in Fig. 3, the stress–strain relation for an undamaged isotropic material may be written in terms of the (undamaged) isotropic elastic stiffness C_{ij} as

$$\{\sigma\} = [C_{ij}]\{\varepsilon\} \tag{8}$$

while, for the ECM of a damaged material, it is written in terms of the damaged (or effective continuum) elastic stiffness $\bar{C}_{ij}$, as

$$\{\sigma\} = [\bar{C}_{ij}]\{\varepsilon\}. \tag{9}$$

In eqns (8) and (9), ε is the elastic components of the strain tensor in which the permanent deformation due to inelastic damage process is assumed to be zero.

In the present work, the undamaged material is assumed to behave isotropically. Nucleation and growth of damage, however, may cause the equivalent damaged material to behave anisotropically. The effective continuum elastic stiffness is to be determined from the undamaged elastic stiffness and a new damage variable $\mathscr{D}$ that reflects the current state (i.e., crack volume fraction, geometry and orientation) using the strain energy equivalence principle.

The strain energy density stored in the ECM subjected to a uniform strain can be written in the form

$$V_{eq} = \tfrac{1}{2}\{\varepsilon\}^{\mathrm{T}}[\bar{C}_{ij}]\{\varepsilon\}. \tag{10}$$

Results described in subsequent sections show that the strain energy density stored in the MVC of a damaged material with a crack under the same boundary strain, can be represented in the form

$$V_d = \tfrac{1}{2}\{\varepsilon\}^{\mathrm{T}}[C_{ij}(1-e_{ij}\mathscr{D})]\{\varepsilon\} \quad \text{(no sum)} \tag{11}$$

where the second term in the bracket, $e_{ij}\mathscr{D}$, is due to the presence of a crack and indicates a loss of strain energy storage capacity comparable to that associated with the damage variable D defined in eqn (5). The parameters e_{ij} in eqn (11) generally depend on the Poisson's ratio of the undamaged material, as well as the crack geometry (shape or aspect ratio) and orientation. This dependence on crack geometry and growth direction can result in locally anisotropic material behavior. The quantity $\mathscr{D}$ in eqn (11) is considered as a new damage variable throughout this paper. Application of the strain energy equivalence principle to eqns (10) and (11) yields the effective continuum elastic stiffness as

$$\bar{C}_{ij} = C_{ij}(1-e_{ij}\mathscr{D}) \quad \text{(no sum)}. \tag{12}$$

In this paper, the strain energy densities for two- and three-dimensional damaged bodies are derived using the strain energy calculations of Kassir and Sih (1967, 1975).

The literature contains a similar, but different, homogenization approach to the prediction of the macroscopic elastic moduli of solid composites. This approach is the self-consistent method of Hershey (1954) and Kröner (1958), originally proposed for aggregates of crystals. In this method, it is usually assumed that the averaged strain (or strain energy) *within* an inclusion embedded in an infinite elastic matrix is approximated in terms of the as-yet-unknown macroscopic elastic moduli of the composite material (Hashin and Rosen, 1964; Hill, 1963, 1965; Budiansky, 1965; Budiansky and O'Connell, 1976). Because the Budiansky and O'Connell (1976) analysis is constrained to yield isotropic moduli for the damaged material, there have been efforts to consider anisotropic crack distributions (Hoeing, 1979; Horri and Nemat-Nasser, 1983; Laws and Dvorak, 1987). However these use the same basic approach as the self-consistent method, namely, to assume the initial crack distribution (isotropic or anisotropic) at a point.

The present method of developing the effective continuum elastic stiffness by considering the change of strain energy storage capacity due to a single crack is thus similar in some respects to the self-consistent method, but differs both in spirit and in some details. Furthermore, in contrast to the present method, isotropic macroscopic elastic moduli obtained using the self-consistent method may not be appropriate for use in a local approach to damage analysis (Lemaitre, 1986), since they are derived by averaging the approximate effects of an initial crack distribution within a RVE.

4.1. *Two-dimensional solid with an elliptical through-crack*

Consider first an infinite two-dimensional elastic solid with an elliptical through-crack under biaxial normal stress (σ_1 and σ_2) and inplane shear (τ_{12}) at infinity. The center of the elliptical crack is located at the origin of the rectangular Cartesian coordinates, with the major axis (length $2a$) and the minor axis (length $2b$) aligned with the coordinates 1 and 2, respectively. In the undamaged state, the total strain energy (per unit thickness) contained in a characteristic region of radius R of the isotropic elastic solid can be expressed as

$$V_0 = \frac{1}{2}\pi R^2 \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{13}$$

where the C_{ij} are the elastic stiffness coefficients of the undamaged solid. For plane stress, these coefficients are defined in terms of the usual engineering constants (Young's modulus E, shear modulus G, and Poisson's ratio ν) as

$$C_{11} = C_{22} = \frac{E}{1-\nu^2} \quad C_{12} = \frac{\nu E}{1-\nu^2} \quad C_{66} = G \tag{14}$$

and, for plane–strain, as

$$C_{11} = C_{22} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \quad C_{12} = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad C_{66} = G. \tag{15}$$

The strain energy released by an elliptical crack was derived in terms of the stresses by Sih and Liebowitz (1967) as

$$V_1 = \frac{1}{2}\pi a^2 \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \left(\frac{1+K}{8G}\right) \begin{bmatrix} (k'+2k'^2) & (-k') & 0 \\ (-k') & (2+k') & 0 \\ 0 & 0 & 2(1+k')^2 \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{16}$$

where $k' = b/a$ is the aspect ratio of the elliptical crack with values $0 \leqslant k' \leqslant 1$. $(1+K)/8G$ is $1/E$ for plane–stress and $(1-\nu^2)/E$ for the plane strain case. Since the stresses in eqn (16) are the stresses at infinite distance from the crack, they may be replaced with the strains at infinity found using the stress–strain relations of the *undamaged* solid. This results in the following expression:

$$V_1 = \frac{1}{2}\pi a^2 \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} C_{11}e_{11} & C_{12}e_{12} & 0 \\ C_{12}e_{12} & C_{22}e_{22} & 0 \\ 0 & 0 & C_{66}e_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{17}$$

where the $e_{ij}(\nu, k')$ are parameters that reflect the physical characteristics and current state of a crack in a solid. Table 2 summarizes expressions for these parameters in two-dimensional planar problems.

Finally, the average strain energy density in a characteristic region of a cracked solid is obtained by subtracting V_1 from V_0 and dividing it by the area πR^2 as:

$$V_d = \frac{1}{2} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 \\ \bar{C}_{12} & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{18}$$

where the $\bar{C}_{ij}$ are found as

$$\begin{aligned} \bar{C}_{11} &= C_{11}(1-e_{11}\mathscr{D}) \quad \bar{C}_{12} = C_{12}(1-e_{12}\mathscr{D}) \\ \bar{C}_{22} &= C_{22}(1-e_{22}\mathscr{D}) \quad \bar{C}_{66} = C_{66}(1-e_{66}\mathscr{D}) \end{aligned} \tag{19}$$

with the damage variable, $\mathscr{D}$, defined as:

$$\mathscr{D} = \left(\frac{a}{R}\right)^2. \tag{20}$$

$\mathscr{D}$ may be interpreted as the ratio of the effective damaged area πa^2 to the total area of the characteristic region considered πR^2, which differs from the damage variable D defined in

Table 2. Values of e_{ij} for plane problems

	Plane stress	Plane strain
e_{11}	$\left(\frac{2\nu^2}{1-\nu^2}\right)+\left(\frac{1-\nu}{1+\nu}\right)k'+\left(\frac{2}{1-\nu^2}\right)k'^2$	$\left(\frac{2\nu^2}{1-2\nu}\right)+(1-2\nu)k'+\left(\frac{2(1-\nu)^2}{1-2\nu}\right)k'^2$
e_{22}	$\left(\frac{2}{1-\nu^2}\right)+\left(\frac{1-\nu}{1+\nu}\right)k'+\left(\frac{2\nu^2}{1-\nu^2}\right)k'^2$	$\left(\frac{2(1-\nu)^2}{1-2\nu}\right)+(1-2\nu)k'+\left(\frac{2\nu^2}{1-2\nu}\right)k'^2$
e_{12}	$\left(\frac{2}{1-\nu^2}\right)-\left(\frac{1-\nu}{\nu(1+\nu)}\right)k'+\left(\frac{2}{1-\nu^2}\right)k'^2$	$\left(\frac{2(1-\nu)^2}{1-2\nu}\right)-\left(\frac{(1-\nu)(1-2\nu)}{\nu}\right)k'+\left(\frac{2(1-\nu)^2}{1-2\nu}\right)k'^2$
e_{66}	$\left(\frac{1}{1+\nu}\right)(1+k')^2$	$(1-\nu)(1+k')^2$

eqn (5), used in typical theories of damage mechanics (e.g., Lemaitre, 1992). Note that, for an infinite plane solid under uniaxial tension in the direction normal to a line crack of length $2a$, the strain energy density of eqn (18) simplifies to

$$V_d = \tfrac{1}{2}E(1-e\mathscr{D})\varepsilon^2 = \tfrac{1}{2}\bar{E}\varepsilon^2 \tag{21}$$

where $e = 2$ for plane stress and $e = 2(1-\nu^2)$ for plane strain. Equation (21) can be also found from Erdogan (1983) by replacing the stresses by the strains at infinity. Erdogan (1983) has shown, from eqn (21), that the crack driving force has the value of $\sigma^2\pi a/E$.

The strain energy equivalence principle states that V_d in eqn (18) should be identical to the strain energy density in eqn (10) for an ECM of a cracked solid, yielding the effective continuum elastic stiffnesses $\bar{C}_{ij}$ as given in eqn (19). One may observe that all the e_{ij} in eqn (19) are positive increasing functions of k' for Poisson's ratios larger than about 0.15 and that, for both the line crack ($k' = 0$) and the circular crack ($k' = 1$), the e_{ij} depend on only the Poisson's ratio. The largest values of e_{ij} are obtained when $k' = 1$, i.e., when the crack is circular. In the case of a circular crack, $\bar{C}_{11} = \bar{C}_{22}$ and $\bar{C}_{66} = (\bar{C}_{11} - \bar{C}_{12})/2$. Thus, a damaged region with a circular microcrack retains isotropic material characteristics.

When k' is not equal to 1, however, eqn (19) and Table 2 indicate that the damaged region behaves orthotropically. In this case, the effective elastic stiffness $\bar{C}_{22}$ is always smaller than $\bar{C}_{11}$. This is consistent with results from fracture mechanics, namely that the presence of the highest stress intensity at the crack edge along the major axis effectively reduces the elastic stiffness in the minor axis direction. In addition, the effective reduction in stiffness associated with the crack may well encourage crack propagation in a direction nearly aligned with the major axis.

In the plane–stress case, the effective continuum elastic moduli can be readily derived, from

$$\begin{aligned}
\bar{E}_{11} &= E\left(\frac{1-\bar{\nu}_{12}\bar{\nu}_{21}}{1-\nu^2}\right)(1-e_{11}\mathscr{D}) \\
\bar{E}_{22} &= E\left(\frac{1-\bar{\nu}_{12}\bar{\nu}_{21}}{1-\nu^2}\right)(1-e_{22}\mathscr{D}) \\
\bar{G}_{12} &= G(1-e_{66}\mathscr{D}) \\
\bar{\nu}_{12} &= \nu\left(\frac{1-e_{12}\mathscr{D}}{1-e_{22}\mathscr{D}}\right) \\
\bar{\nu}_{21} &= \nu\left(\frac{1-e_{12}\mathscr{D}}{1-e_{11}\mathscr{D}}\right).
\end{aligned} \tag{22}$$

Thus, if the elastic moduli of an undamaged material (E, G, ν) are given, and the damage $(k', \mathscr{D}$, orientation) is known, the effective elastic properties of the damaged material $(e_{ij}; \bar{\nu}_{12}, \bar{\nu}_{21}, \bar{G}_{12}, \bar{E}_{11}, \bar{E}_{22})$ may be found.

As mentioned in the preceding, experiments have shown that materials typically fail at a critical value of D in eqn (5) that is less than 1. Since the strain energy density, V_d, stored in a cracked solid must be non-negative, $\mathscr{D}_c$, a critical value of $\mathscr{D}$, may be determined from the condition that the determinant of the effective continuum elastic stiffness matrix is zero at that value:

$$\det[\bar{C}_{ij}(C_{ij}, e_{ij}; \mathscr{D}_c)] = 0. \tag{23}$$

The condition (23), which is derived herein from the positive condition of strain energy density, looks quite similar to that of limit point bifurcation in materials (see Neilsen and

Schreyer, 1993). But, clarifying the relation between this critical damage condition and the limit point bifurcation condition may require continued consideration.

Some extreme critical values may be obtained for the plane stress case as:

$$\mathscr{D}_c = \begin{cases} \dfrac{1-\nu^2}{2} & \text{when } k' = 0 \\ \dfrac{1+\nu}{4} & \text{when } k' = 1 \end{cases} \tag{24}$$

and, for the plane strain case, as

$$\mathscr{D}_c = \begin{cases} \dfrac{1-2\nu}{2(1-\nu)^2} & \text{when } k' = 0 \\ \dfrac{1-2\nu}{2(1-\nu)} & \text{when } k' = 1. \end{cases} \tag{25}$$

In general, the critical value for plane strain is lower than that for plane stress. In the special case of an elastic plane solid under uniaxial tension in the direction normal to a through line crack, the specific strain energy for the cracked solid is given by eqn (21), and yields $\mathscr{D}_c = 0.5$. The same critical value was developed by Krajcinovic and Silva (1982) using a parallel bar model.

4.2. *Three-dimensional solid with an embedded elliptical plane crack*

Now, consider an infinite three-dimensional elastic solid with an embedded elliptical plane crack under general loading. The origin of a rectangular Cartesian coordinate system (crack coordinates) is located at the center of the ellipse. The coordinate directions 1 and 2 are aligned with the major axis (length $2a$) and the minor axis (length $2b$) of the crack, respectively, while the coordinate 3 is normal to the plane of the crack.

Under general loading, the undamaged isotropic elastic body has strain energy density given by

$$V_0 = \frac{1}{2}\left(\frac{4\pi R^3}{3}\right)\left(\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}^T \begin{bmatrix} C_{11} & C_{12} & C_{12} \\ C_{12} & C_{11} & C_{12} \\ C_{12} & C_{12} & C_{11} \end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} + \begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}^T \begin{bmatrix} C_{66} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}\begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}\right) \tag{26}$$

where the C_{ij} are the elastic stiffness coefficients given by eqn (15). The strain energy released by an elliptical surface crack embedded in a three-dimensional solid under general loading was derived by Kassir and Sih (1967), in terms of the applied stresses at infinity, as

$$V_1 = \frac{4\pi ab^2}{3E(k)}\left(\frac{1-\nu^2}{E}\right)\sigma_3^2 + \frac{2\pi ab^2}{3}\left(\frac{1-\nu}{G}\right)[X(k,\nu)\tau_{31}^2 + Y(k,\nu)\tau_{23}^2] \tag{27}$$

in which

$$X(k,\nu) = \frac{k^2}{(k^2-\nu)E(k)+\nu k'^2 K(\nu)}$$

$$Y(k,\nu) = \frac{k^2}{(k^2+\nu k'^2)E(k)-\nu k'^2 K(k)}$$

$$k^2 = 1-k'^2. \tag{28}$$

In the preceding equations, $E(k)$ and $K(k)$ are the complete elliptic integrals of the first and second kind, respectively. Since the stresses in eqn (27) are applied far from the crack, they may be replaced by the equivalent strain at infinity using the stress–strain relations of the undamaged isotropic solid. This yields

$$V_1 = \frac{1}{2}\left(\frac{4\pi R^3}{3}\right)\left(\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}C_{11}e_{11}\mathscr{D} & C_{12}e_{12}\mathscr{D} & C_{12}e_{13}\mathscr{D}\\ C_{12}e_{12}\mathscr{D} & C_{11}e_{22}\mathscr{D} & C_{12}e_{23}\mathscr{D}\\ C_{12}e_{13}\mathscr{D} & C_{12}e_{23}\mathscr{D} & C_{11}e_{33}\mathscr{D}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} + \begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}C_{66}e_{44}\mathscr{D} & 0 & 0\\ 0 & C_{66}e_{55}\mathscr{D} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}\right) \tag{29}$$

where

$$e_{11} = e_{22} = \left(\frac{2v^2}{1-2v}\right)H(k) \qquad e_{33} = \left(\frac{2(1-v)^2}{1-2v}\right)H(k)$$

$$e_{12} = \left(\frac{2v(1-v)}{1-2v}\right)H(k) \qquad e_{13} = e_{23} = e_{33}$$

$$e_{44} = \frac{(1-v)k^2H(k)}{(k^2+vk'^2)-vH(k)K(k)} \qquad e_{55} = \frac{(1-v)k^2H(k)}{(k^2-v)+vH(k)K(k)}$$

$$H(k) = \frac{k'^2}{E(k)}. \tag{30}$$

Although Kassir and Sih (1967) did not include the shear stress τ_{12} at infinity in their analysis, it is accommodated here under the assumption that it does not change their original calculation of V_1 significantly. (This assumption may require continued consideration.) In eqn (29), the damage variable, $\mathscr{D}$, represents the ratio of the effective damaged volume to the total volume of the MVC considered:

$$\mathscr{D} = \left(\frac{a}{R}\right)^3. \tag{31}$$

Subtracting V_1 (eqn (29)) from V_0 (eqn (26)), and dividing by the volume, the strain energy density stored in the cracked solid is found as:

$$V_d = \frac{1}{2}\left(\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13}\\ \bar{C}_{12} & \bar{C}_{11} & \bar{C}_{13}\\ \bar{C}_{13} & \bar{C}_{13} & \bar{C}_{33}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} + \begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar{C}_{44} & 0 & 0\\ 0 & \bar{C}_{55} & 0\\ 0 & 0 & \bar{C}_{66}\end{bmatrix}\begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}\right) \tag{32}$$

where the $\bar{C}_{ij}$ are given by

$$\bar{C}_{11} = C_{11}(1-e_{11}\mathscr{D}) \quad \bar{C}_{33} = C_{11}(1-e_{33}\mathscr{D})$$

$$\bar{C}_{12} = C_{12}(1-e_{12}\mathscr{D}) \quad C_{13} = C_{12}(1-e_{13}\mathscr{D})$$

$$\bar{C}_{44} = C_{66}(1-e_{44}\mathscr{D}) \quad \bar{C}_{55} = C_{66}(1-e_{55}\mathscr{D})$$

$$\bar{C}_{66} = C_{66}. \tag{33}$$

As in the case of the two-dimensional solid, the strain energy equivalence principle requires that the average strain energy density of the cracked MVE be the same as the strain energy density of the equivalent continuum model of the cracked solid. Again, this implies that the effective continuum elastic stiffness for the equivalent continuum model is given by the $\bar{C}_{ij}$ of eqn (33). Clearly, material in the three-dimensional body with an embedded elliptical plane crack behaves orthotropically, as expected. Also observe that, since e_{33} is always larger than e_{11} or e_{22}, the effective continuum elastic stiffness $\bar{C}_{33}$ in the direction normal to the crack surface is the softest. For a penny-shaped (circular) crack, $\bar{C}_{44}$ becomes identical to $\bar{C}_{55}$, resulting in transversely isotropic behavior. Furthermore, in this case, the e_{ij} have the largest values, which depend only on the Poisson's ratio. In this case, they are found as:

$$e_{11} = e_{22} = \frac{4}{\pi}\left(\frac{v^2}{1-2v}\right) \quad e_{33} = \frac{4}{\pi}\left(\frac{(1-v)^2}{1-2v}\right)$$

$$e_{12} = \frac{4}{\pi}\left(\frac{v(1-v)}{1-2v}\right) \qquad e_{13} = e_{23} = e_{33}$$

$$e_{44} = e_{55} = \frac{4}{\pi}\left(\frac{1-v}{2-v}\right). \tag{34}$$

5. LOCAL DAMAGE CONSIDERED AS AN EQUIVALENT ELLIPTICAL MICROCRACK

The elastic behavior of a cracked solid was considered in the preceding section, leading to effective continuum elastic stiffness $\bar{C}_{ij}$. In effect, as shown in eqns (19) and (33), a solid with embedded microcracks is smeared smoothly into an equivalent continuum with elastic stiffness $\bar{C}_{ij}$. By replacing all MVC in which microcracks exist with the equivalent anisotropic (for elliptical cracks, orthotropic) properties, conventional methods may be employed for further stress and damage analyses. Also, as noted out by Lemaitre (1986), in this continuum concept, a crack edge may be considered as a local process zone in which damage increases until complete local failure of material occurs. This may perhaps be considered a continuous (regularized) version of crack propagation.

This local approach seems useful only when the current state of microcracks (geometry and growth direction) are known, as that information is required to calculate the effective continuum elastic stiffness. In practice, it is perhaps impossible to identify this state in detail. A certain level of damage associated with microdefects exists even before microcracks with well-defined surfaces appear. Thus, a damage state smaller than the critical value only implies a degradation of material properties through softening, not necessarily visible cracks. Unfortunately, for internal microcracks, the current damage state (geometry and growth direction) is inaccessible. Hence, prediction of damage evolution and fracture using a continuum approach requires a method by which the damage state can be determined at some time. In addition, there should be a relation with which current damage information can be converted to an effective continuum representation, that is, eqn (19) or (33).

As can be seen from eqns (19) and (33), local damage associated with an elliptical (non-circular) microcrack within a MVC always changes the virgin isotropic properties into orthotropic properties. Thus, an observed change from isotropic to orthotropic behavior may imply the existence of local damage. Furthermore, based on this reasoning, it may be appropriate to assume that a general damage state can be approximately represented as an equivalent elliptical microcrack. In other words, an elliptical microcrack may be considered as a construct which relates some general damage to the effective material properties $\bar{C}_{ij}$ of an ECM. Figure 4 illustrates the connections between the several related ways of thinking about local damage: (1) general local damage; (2) an equivalent elliptical crack (damage density, aspect ratio, growth direction (orientation)); and (3) the effective continuum elastic stiffness.

To capture the effects of local damage in the $\bar{C}_{ij}$, the geometry and orientation of the equivalent elliptical microcrack are required. Since stress analysis is typically conducted in

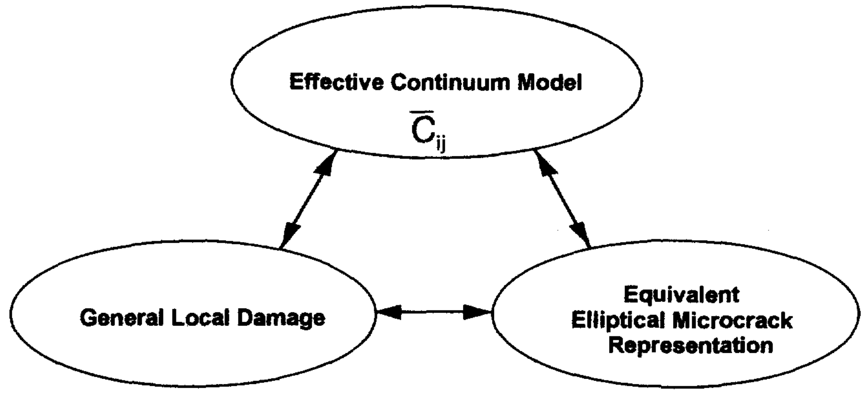

Fig. 4. Relation between general local damage, an equivalent elliptical microcrack, and the effective continuum model.

the course of damage analysis, stress analysis results are assumed to be available for use in determining the geometry (aspect ratio) and growth direction (crack coordinates) of the equivalent elliptical microcrack. The following section addresses this idea in more detail.

6. DETERMINATION OF EQUIVALENT MICROCRACK CHARACTERISTICS

In the continuum sense, local damage associated with any microdefect may be considered as an equivalent (fictitious) elliptical microcrack so that eqn (19) or (33) may be used to represent it in terms of the effective continuum elastic stiffness. As seen in these equations, damage associated with an elliptical crack is defined by the crack coordinate directions, the aspect ratio $k' = b/a$, and the scalar damage variable $\mathscr{D}$. As the evolution of the damage variable $\mathscr{D}$ is considered in the following section, the present discussion is confined to the two geometrical parameters: crack coordinate directions and aspect ratio. In this damage modeling approach, the likely direction of damage propagation depends somewhat on these parameters, that is, damage propagation is somewhat aligned with the softest (most damaged) direction.

The undamaged virgin material is assumed to be isotropic. With damage, considering eqns (19) and (33), this is no longer generally so, except for cracks with aspect ratios of 1. In general, aspect ratios have values between 0 and 1. Since eqns (19) and (33) were developed with respect to the crack coordinates, the orientation of the crack relative to the global structural coordinates must be determined. The effective continuum elastic stiffness $\bar{C}_{ij}$ can be determined with respect to the crack coordinates, then, knowing the local orientation and aspect ratio, can be transformed back to the global structural coordinates and used in the next step of an incremental calculation process. Thus, the determination of crack coordinates and aspect ratio of an equivalent elliptical microcrack is essential for the prediction of damage propagation.

In the fracture mechanics and damage mechanics literature, there are numerous papers on subjects related to crack propagation. Thorough reviews can be found in the paper by Krajcinovic (1989) and in the recent book by Sih (1991). In the present work, a new method is proposed, based on determining the principal coordinates and aspect ratio of an equivalent elliptical microcrack, features which influence the continuum crack growth direction.

In the course of an incremental damage analysis, the current values of stresses and strains at a damaged local point are assumed to be available. Determination of the crack coordinate directions and the crack aspect ratio from this information is desired.

The first of these problems benefits from the assumption of elliptical microcracks, in that the associated damage results in local orthotropic material behavior. That is, the orthotropic material principal coordinates may be considered to be aligned with the crack coordinates. The principal stress or principal strain directions are possible choices for this orientation because of their natural biaxial nature. However, since the principal stress and principal strain directions in an orthotropic material are not necessarily aligned, and because there is no clear reason to choose one or the other, some alternate, perhaps approximate method may be used. Such an alternate orientation is that which minimizes the strain energy associated with shear deformation. Once the principal stresses and strains are known, the maximum and minimum shear stresses and strains may also be readily determined.

In the following two sub-sections, the determination of crack coordinates and aspect ratio is addressed, first for two-dimensional damage and then for three-dimensional damage.

6.1. *Two-dimensional damage*

As discussed in the preceding, alternate approaches may be used to determine the principal material directions of the orthotropic damaged material. These include principal stress, principal strain, an orientation that minimizes strain energy associated with shear, or an orientation that minimizes the strain energy associated with stiffness components C_{13} and C_{23}.

Assuming that the crack orientation has been established to within a rotation of 90 degrees using one of these approaches, the "1" and "2" directions remain to be determined. Given the stresses and strains referred to the global structural coordinate system, the stresses and strains in the crack coordinate system (I, II) may be readily found. Furthermore, the material moduli (inverse compliances, $1/S_{\mathrm{I,I}}$ and $1/S_{\mathrm{II,II}}$) associated with uniaxial stresses in each of the orthogonal directions which define the crack coordinate system may be determined. If these are identical, the material is locally isotropic, either having no local damage or circular microcracks. Otherwise, the direction associated with the highest stiffness corresponds to the "1" direction, the major axis of the elliptical crack.

From eqns (9) and (12), the shear stress–strain relation may be used to establish the current aspect ratio k'. Assuming that the stresses and strains in the crack coordinate system approximate the principal stresses and strains, the effective maximum shear stress and strain are estimated as:

$$\tau = \frac{|\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}|}{2} \quad \gamma = \frac{|\varepsilon_{\mathrm{I}} - \varepsilon_{\mathrm{II}}|}{2}. \tag{35}$$

These yield an estimate for the effective elastic shear stiffness:

$$G^* = \frac{\tau}{\gamma}. \tag{36}$$

Assuming the computed value of G^* is identical to the effective continuum shear stiffness of eqn (12), the aspect ratio k' may be computed using

$$k' = 1 - \left[\frac{1}{\alpha\mathscr{D}}\left(1 - \frac{G}{G^*}\right)\right]^{1/2} \tag{37}$$

where $\alpha = 1/(1+\nu)$ for plane stress and $\alpha = (1-\nu)$ for plane strain. In general, $G/G^* \leqslant 1$ since G^* is softened from the virgin value of G.

6.2. *Three-dimensional damage*

As discussed in the preceding, alternate approaches may be used to determine the principal material directions of the orthotropic damaged material. These include principal stress, principal strain, or an orientation that minimizes strain energy associated with shear.

A similar approach is used to address three-dimensional damage. The crack coordinate directions with respect to global coordinates may be established by considering principal stress, principal strain, or an orientation that minimizes strain energy associated with shear. Then, stresses and strains in that coordinate system may be calculated and, by considering material moduli under uniaxial stress, the "1", "2", and "3" directions may be distinguished.

The "softest" direction under uniaxial normal stress corresponds to the normal to the planar crack, the "3" direction.

The remaining problem is determining how to match the two principal material directions I and II to the crack coordinates 1 and 2. From eqn (33), the effective elastic shear stiffnesses $\bar{C}_{44}$ and $\bar{C}_{55}$ can be used to distinguish the major axis of the elliptical surface crack from the minor axis; $\bar{C}_{11}$ and $\bar{C}_{22}$ are not useful for this purpose since they are identical. From eqn (33), however, $\bar{C}_{55}$ is clearly softer than $\bar{C}_{44}$ for any $0 \leqslant k' \leqslant 1$, and both are always softer than $\bar{C}_{66} = G$, the stiffness of the virgin material. Thus the "stiffest" direction under uniaxial shear stress corresponds to the "1" direction.

From the stresses and strains in the principal material coordinate system, the approximate maximum or minimum shear stresses and strains may be estimated as:

$$\tau_{\mathrm{II,III}} = \frac{|\sigma_{\mathrm{II}} - \sigma_{\mathrm{III}}|}{2} \quad \gamma_{\mathrm{II,III}} = \frac{|\varepsilon_{\mathrm{II}} - \varepsilon_{\mathrm{III}}|}{2}$$
$$\tau_{\mathrm{I,III}} = \frac{|\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}|}{2} \quad \gamma_{\mathrm{I,III}} = \frac{|\varepsilon_{I} - \varepsilon_{\mathrm{III}}|}{2}. \tag{38}$$

The current local effective elastic shear stiffnesses are now estimated approximately from

$$G^*_{44} = \frac{\tau_{\mathrm{II,III}}}{\gamma_{\mathrm{II,III}}} \quad G^*_{55} = \frac{\tau_{\mathrm{I,III}}}{\gamma_{\mathrm{I,III}}}. \tag{39}$$

By comparing these estimated effective shear stiffnesses, the major axis of the equivalent elliptical crack, 1, is seen to correspond to the principal stress direction I if $G^*_{55} < G^*_{44}$, and vice versa.

To predict the current aspect ratio k', the estimated values of G^*_{44} and G^*_{55} are assumed to be identical to $\bar{C}_{44}$ and $\bar{C}_{55}$, respectively, to obtain

$$G^*_{44} = G[1 - e_{44}(\nu, k')\mathscr{D}]$$
$$G^*_{55} = G[1 - e_{55}(\nu, k')\mathscr{D}] \tag{40}$$

where e_{44} and e_{55} are defined in eqn (30). Since e_{55} is always larger than e_{44} for any Poisson ratio $\nu \leqslant 0.5$ and aspect ratio $0 \leqslant k' \leqslant 1$, $G^*_{55} < G^*_{44}$ is confirmed. Note that both e_{44} and e_{55} have the same maximum value of $4/\pi(1-\nu)$ when $k' = 1$, which does not exceed one.

Finally, the aspect ratio k' for a given current damage $\mathscr{D}$ may be readily estimated using

$$e_{55}(\nu, k') - e_{44}(\nu, k')g^* = \frac{1}{\mathscr{D}}(1 - g^*) \tag{41}$$

where

$$g^* = \frac{G^*_{55}}{G^*_{44}}. \tag{42}$$

7. CONSISTENT DAMAGE EVOLUTION EQUATION

In the preceding sections, the orthotropic effective continuum elastic stiffness representation of local damage due to microdefects was introduced using a single scalar damage variable $\mathscr{D}$. This damage variable $\mathscr{D}$ is defined, in eqns (20) and (31), as

$$\mathscr{D} = \left(\frac{a}{R}\right)^n \tag{43}$$

where $n = 2$ for the through elliptical microcrack in the two-dimensional solid and $n = 3$ for the embedded planar elliptical microcrack in the three-dimensional solid. Herein the definition given by eqn (43) is used to develop a consistent damage evolution equation. Differentiation of eqn (43) with respect to time yields

$$\frac{\dot{\mathscr{D}}}{\mathscr{D}} = n\frac{\dot{a}}{a}. \tag{44}$$

Using eqn (43), the semi-major axis of the crack a in eqn (44) can be eliminated to find

$$\dot{\mathscr{D}} \propto \mathscr{D}^{[1-(1/n)]}\dot{a}. \tag{45}$$

Since the microcrack size, a, is not available in general, the crack growth rate, $\dot{a}$, should be expressed in terms of measurable or predictable quantities. In fracture mechanics, experiments have shown that slow crack growth is related to the stress intensity factor according to Paris' Law (Paris and Erdogan, 1963). That is

$$\dot{a} = AK_{\mathrm{I}}^{\mathrm{N}} \quad K_I = Y\sigma\sqrt{a} \tag{46}$$

where A and N are parameters to be determined from experiments, and K_{I} is the stress intensity factor, which depends on crack length, applied stress and a geometrical factor Y. From eqns (43) and (46), rate of microcrack growth is found to be

$$\dot{a} \propto \sigma^N a^{N/2} \propto \sigma^N \mathscr{D}^{N/2n}. \tag{47}$$

Combining the relations (45) and (47) yields

$$\dot{\mathscr{D}} \propto \mathscr{D}^{[1-(1/N)+(N/2n)]}\sigma^N. \tag{48}$$

In the sense of Paris' law, the stress σ in eqn (48) should be the stress applied normal to the mid-plane between two microcrack surfaces. Therefore the stress σ_2 may be used in eqn (48) for two-dimensional damage, and the stress σ_3 for three-dimensional damage. A damage threshold may be introduced, by using the Heaviside step function $H(\sigma_{eq} - \sigma_{TH})$, as

$$\dot{\mathscr{D}} = \beta\mathscr{D}^{[1-(1/n)+(N/2n)]}\sigma_n^N H(\sigma_{eq} - \sigma_{TH}) \tag{49}$$

where $\sigma_n = \sigma_2$ for two-dimensional damage and $\sigma_n = \sigma_3$ for three-dimensional damage. In eqn (49), β is a material constant to be determined from experiments, and σ_{TH} is a threshold stress above which damage will grow. To accommodate general three dimensional stress states, the driving stress in eqn (49) has been replaced by σ_{eq}, the von Mises equivalent stress calculated from the deviatoric stresses, as:

$$\sigma_{eq} = \left(\frac{3}{2}\sigma_{ij}^D \sigma_{ij}^D\right)^{1/2}. \tag{50}$$

Alternatively, the so-called damage equivalence stress may be used in place of the von Mises equivalent stress (Lemaitre, 1992), defined as

$$\bar{\sigma} = \sigma_{eq} R^{1/2} \tag{51}$$

where

$$R = \frac{2}{3}(1+\nu)+3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2 \tag{52}$$

and σ_H denotes the hydrostatic stress. In order to determine the parameters β and σ_{TH} in eqn (49), an experimental approach similar to that of Lemaitre (1992) may be followed.

Neither the von Mises stress nor the damage equivalence stress may be ideal for use in this application, as both were developed for use with isotropic materials. An alternative for future research might involve a "deformation-based" (i.e., strain) damage growth criterion, perhaps driven by a strain energy density level.

8. ITERATIVE NUMERICAL METHOD FOR DAMAGE ANALYSIS

The previous sections outlined a method by which to estimate current equivalent microcrack characteristics from the current values of stresses and strains at a damaged local point. The current stresses and strains should be related through the current values of effective continuum elastic stiffness $\bar{C}_{ij}(t)$, which depends on time-varying equivalent microcrack characteristics. Therefore, in contrast to the case of undamaged isotropic state, a special numerical strategy may be required to enforce this condition during a damage evolution simulation. One possible approach is described in what follows.

In this approach, the effective continuum elastic stiffness at the previous time t_0, $\bar{C}_{ij}(t_0)$, is used in the structural analysis during the current time step to yield initial estimates of current stresses $\sigma_i(t)$ and strains $\varepsilon_j(t)$ at current time $t=(t_0+\mathrm{d}t)$ from the relation

$$\sigma_i(t) \cong \bar{C}_{ij}(t_0)\varepsilon_j(t). \tag{53}$$

Also, the current value of damage variable $\mathscr{D}(t)$ may be calculated from the damage evolution equation (49) using the previous values of $\mathscr{D}(t_0)$ and $\sigma_i(t_0)$ and a finite difference numerical scheme as:

$$\mathscr{D}(t) \cong \mathscr{D}(t_0)+\mathrm{d}t\beta\mathscr{D}(t_0)^{[1-(1/n)+(N/2n)]}\sigma_n^N(t_0)H(\sigma_{eq}(t_0)-\sigma_{TH}). \tag{54}$$

Since the current value of $\mathscr{D}(t)$ is known, as well as the initial estimates of $\sigma_i(t)$ and $\varepsilon_j(t)$, initial estimates of crack coordinate directions θ_i and aspect ratio k' may be obtained using the method described in Section 6. Then, an initial estimate of the effective continuum elastic stiffness $\bar{C}_{ij}(t)$ may be calculated. Iteration may be used to improve the accuracy of this estimate.

The new estimate of $\bar{C}_{ij}(t)$ may be used as a starting point for repeated calculation of structural response during the current time step, leading to improved estimates of stress, strain, damage, crack coordinate directions, and crack aspect ratio. These are subsequently used to improve the estimate of the effective continuum elastic stiffness. The process repeats at each time step until satisfactorily converged.

Though a specific iterative numerical approach is described herein, there are undoubtedly several alternate numerical approaches to predicting the current equivalent microcrack characteristics in consistent ways. These issues are the subjects of continuing research.

9. CONCLUSIONS

To accommodate damage associated with general microdefects and to model the progress of local fracture in a continuum sense, the concept of an equivalent elliptical microcrack representation of local damage was introduced and explored. A strain energy equivalence principle (SEEP) was developed based on the idea that, under the same strain, an effective continuum model of a damaged material should store the same strain energy as the damaged material. A scalar damage variable with a physical interpretation as a crack volume fraction was used to develop a consistent damage evolution equation; this equation does not, however, address damage nucleation. The SEEP was used to derive the effective continuum elastic stiffness of two-dimensional and three-dimensional cracked solids. A damage evolution equation was developed based on a crack growth law commonly used in fracture mechanics. These features may be combined in an iterative incremental stress analysis to model damage development. The combination of damage variable and damage evolution equation permits an initially isotropic material to behave orthotropically and may provide a basis for a continuum approach to crack propagation analysis.

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